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Calculation of Coastally Trapped Waves

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Contents

Chapter 1

Introduction

1.1 Motivation

We concern ourselves, as in Schmidt and Johnson[6], with the energy of low frequency motions in the shallow seas and at the borders of oceans. Most of the energy involved in these motions is concentrated in coastally trapped waves which arise in two ways. Stratification in a flow field along with a flat bottom give rise to baroclinic internal Kelvin waves, while a sloping shelf without any stratification gives rise to barotropic shelf waves. When both mechanisms are considered, the resulting waves are a combination of the two above.

For the continuously stratified case, there exists a countably infinite set of discrete modes with phase speeds that decrease to zero, forming a complete set. Each of these modes travels in the same direction. As this is a complete set, any pressure field is expressible as an infinite series of these modes. The fundamental mode is essentially an external Kelvin wave and has the maximum phase speed.

As there is no analytic solution to this problem, we concern ourselves with a new numerical scheme that will allow us to calculate the modes and their pressure fields.

Analytic solutions are available in extremely limited cases, and we may use these to test the accuracy of our numerical method. There are numerical methods available, for example the suite of programs developed by Brink and Chapman[2], but these are highly outdated and solve the method across the entire two dimensional domain. More recently, work has been completed in this area by Johnson and Rodney[4], who showed that the boundary condition imposed by Brink and Chapman[2] limited the overall accuracy of their method and could not be improved upon using their methods. In particular, their work was made more efficient in the low frequency limit at the request of their referee, Brink, as he stated that this limit is the most important regime. This low frequency limit is the same as that which we consider in this paper. Again, however, this method solved the problem across the entire two dimensional domain. The dynamics of the problem relies entirely on the shelf profile and the stratification, therefore we attempt as in Schmidt and Johnson[6] to reduce the problem to a one dimensional problem along the shelf. The limitations of the method proposed by Schmidt and Johnson[6] will be addressed as their method only remains valid for uniform stratification, we attempt to extend the method to non uniform stratification.

1.2 Equations of Motion

We consider a semi-infinite ocean of depth D bounded by a monotonic sloping shelf with width of order L. We take Cartesian coordinates (x^*, y^*, z^*), with x along the shelf, y out to sea and z vertically upwards, where the shelf profile depends on y alone (or can be considered to depend on z alone, so long as it is uniform in x). We take the flow to be Boussinesq and incompressible with uniform Coriolis frequency f, total density $\rho_0^*(z^*) + \rho^*(\mathbf{x}^*, t)$ and pressure $p_0^*(z^*) + p^*(\mathbf{x}^*, t)$.

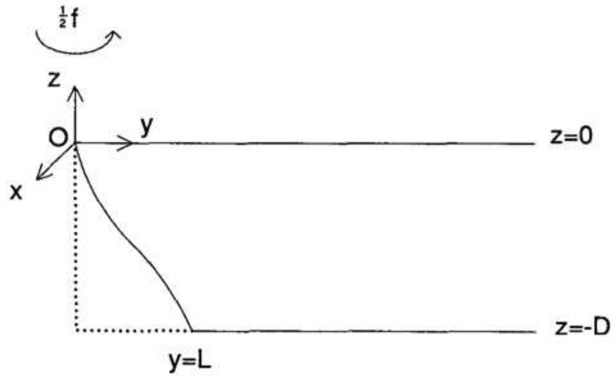

Figure 1.1: Domain

We introduce the buoyancy frequency:

$$N_0^2 N^2(z^*) = -\frac{g}{\rho_0^*(0)} \frac{d\rho_0^*}{dz^*},$$

where we choose N_0 so that the maximum value of $N(z^*)$ is unity. We consider waves of frequency ωf and introduce the scalings:

$$\begin{aligned}
(x, y, z) &= \left(\frac{x^*}{L}, \frac{y^*}{L}, \frac{z^*}{D} \right) \\
(u, v, w) &= \left(\frac{u^*}{U}, \frac{v^*}{U}, \frac{w^* N_0^2 D}{\omega f^2 U L} \right) \\
p &= \frac{p^*}{\rho_0^*(0) f U L} \\
\rho &= \frac{g D \rho^*}{\rho_0^*(0) f U L}.
\end{aligned}$$

The linearised equations for a Boussinesq flow in a rotating reference frame are:

$$\begin{aligned}
\frac{\partial u^*}{\partial t} - f v^* &= -\frac{1}{\rho_0^*(0)} \frac{\partial p^*}{\partial x^*} \\
\frac{\partial v^*}{\partial t} + f u^* &= -\frac{1}{\rho_0^*(0)} \frac{\partial p^*}{\partial y^*} \\
\frac{\partial w^*}{\partial t} &= -\frac{1}{\rho_0^*(0)} \frac{\partial p^*}{\partial z^*} - \frac{\rho^* g}{\rho_0^*(0)} \\
\frac{\partial}{\partial t} \left(-\frac{\rho^* g}{\rho_0^*(0)} \right) &+ N_0^2 N^2 w^* = 0
\end{aligned}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0.$$

Nondimensionalising, using the scalings above, we find that:

$$\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -f\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + fu &= -f\frac{\partial p}{\partial y} \\
\frac{\omega f}{N_0^2}\frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial z} - \rho \\
f\frac{\partial \rho}{\partial t} + \omega f^2 N^2 w &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\omega f^2 L^2}{N_0^2 D^2}\frac{\partial w}{\partial z} &= 0.
\end{aligned}$$

Now considering waves of frequency ωf propagating in the positive x-direction with wavenumber k, i.e. considering:

$$(u, v, w, \rho, p) = (u(y, z), v(y, z), w(y, z), \rho(y, z), p(y, z))e^{i(\omega f t - kx)}$$

and noting that $\partial/\partial t$ is equivalent to multiplication by $i\omega f$ and $\partial/\partial x$ is equivalent to multiplication by $-ik$, when we cancel the common factor of $e^{i(\omega f t - kx)}$, we find that:

$$i\omega u - v = ikp \tag{1.1}$$

$$i\omega v + u = -p_y \tag{1.2}$$

$$i\left(\frac{\omega f}{N_0}\right)^2 w = -p_z - \rho \tag{1.3}$$

$$i\rho - N^2 w = 0 \tag{1.4}$$

$$-iku + v_y + \frac{\omega}{B^2} w_z = 0, \tag{1.5}$$

where $B = N_0 D/fL$ measures the importance of stratification. Note also that we have adopted the subscript notation to denote partial derivatives.

We combine equations (1.1) and (1.2) to give the velocity components in terms of p alone:

$$u = \frac{-p_y + \omega k p}{1-\omega^2} \tag{1.6}$$

$$v = \frac{-i(kp + \omega p_y)}{1-\omega^2} \tag{1.7}$$

Assuming that $N_0 \gg \omega f$, i.e. the incident frequency is small compared to the buoyancy frequency N_0, we find from equation (1.3) that the flow is hydrostatic, with:

$$\rho = -p_z \tag{1.8}$$

And from (1.4) along with (1.8):

$$w = \frac{ip_z}{N^2} \tag{1.9}$$

Substituting equations (1.6), (1.7) and (1.9) into the continuity equations (1.5) gives the governing equations for p:

$$-k^2 p + p_{yy} + (1-\omega^2)B^{-2}(N^{-2}p_z)_z = 0 \tag{1.10}$$

The boundary condition at the free surface, $z = 0$, is then:

$$a^2 p_z + B^2 N^2(0) p = 0 \quad (\text{on } z = 0)$$

Where $a = (gD)^{1/2}/fL$ is the Rossby radius of deformation. If we take $z = -h(y)$, for $0 \leq y \leq 1$, to be the shelf profile (or as we will take later on, $y = d(z)$), then we have, from the fact that the normal velocity vanishes on a solid boundary, that:

$$\left(\frac{\omega}{B^2}\right) w = -vh' \quad (\text{on } z = -h(y)), \text{ and}$$

$$w = 0 \quad (\text{on } z = -1,\ y > 1)$$

Substituting in our equations for v and w in terms of p, i.e. equations (1.7) and (1.9), we find that:

$$(1 - \omega^2)cp_z = -h' B^2 N^2 (p + cp_y) \quad (\text{on } z = -h(y))$$
$$p_z = 0 \quad (\text{on } z = -1,\ y > 1)$$

Where $c = \omega/k$ is the wave speed. If we consider only long-wave solutions, as this is where most of the energy of these motions is contained, we take the limit $\omega \to 0$, $k \to 0$ with c remaining fixed. Taking this limit, we find that the waves are non-dispersive, satisfying:

$$p_{yy} + B^{-2}(N^{-2}p_z)_z = 0, \tag{1.11}$$

under the conditions:

$$h' N^2 B^2 (p + cp_y) + cp_z = 0 \quad (\text{on } z = -h(y),\ y \leq 1) \tag{1.12}$$
$$a^2 p_z + B^2 N^2(0) p = 0 \quad (\text{on } z = 0) \tag{1.13}$$
$$p_z = 0 \quad (\text{on } z = -1,\ y > 1). \tag{1.14}$$

We now introduce a new length scale based on the strength of the stratification $Y = y/B$, where $B = N_0 D/f$. Then the governing equation for p, (1.11) becomes:

$$p_{YY} + (N^{-2}p_z)_z = 0, \tag{1.15}$$

under the conditions:

$$B^{-1} K d'(z) p_z = N^2 (p + K p_Y) \quad (\text{on } Y = B^{-1} d(z)) \tag{1.16}$$

$$A^2 P_z + p = 0 \quad (\text{on } z = 0) \tag{1.17}$$

$$p_z = 0 \quad (\text{on } z = -1,\ y > 1), \tag{1.18}$$

where $K = B^{-1}c$ and $A = a/(BN(0))$. We now perform separation of variables on (1.15), by looking for solutions of the form $p = \sum_{n=0}^{\infty} \widetilde{Y}_n(Y) Z_n(z)$, which leads us to:

$$\frac{\widetilde{Y}_n''}{\widetilde{Y}_n} = -\frac{(N^{-2}Z_n')'}{Z_n} = \mu_n \text{ (constant)}.$$

So we have that the vertical structure has eigenfunctions, Z_n, and eigenvalues, λ_n satisfying the Sturm-Liouville problem:

$$(N^{-2}Z_n')' + \lambda_n^2 Z_n = 0, \tag{1.19}$$

under the boundary conditions:

$$Z_n' = 0 \quad (\text{on } z = 1, y > 1) \tag{1.20}$$

$$A^2 Z_n' + Z_n = 0 \quad (\text{on } z = 0). \tag{1.21}$$

And we therefore find that the horizontal structure satisfies:

$$\widetilde{Y}_n'' - \lambda_n^2 \widetilde{Y} = 0.$$

We need $\widetilde{Y}$ to decay (or rather to become constant) as $y \to \infty$, and we find that:

$$\widetilde{Y} = A_n e^{-\lambda_n Y}$$

We then have a separated solution for p in the form:

$$p(Y, z) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n Y} Z_n(z), \tag{1.22}$$

where we find the coefficients a_n by substituting this into condition (1.16):

$$N^2(z) \sum_{n=0}^{\infty} a_n(1 - K\lambda_n) e^{-\lambda_n d(z)/B} Z_n(z) = B^{-1} K d' \sum_{n=0}^{\infty} a_n e^{-\lambda_n d(z)/B} Z_n'(z) \tag{1.23}$$

Chapter 2

Numerical Method Using Spectral Differentiation

This method has an advantage over transforming the equation into an integral form in that it can more readily be used to solve the problem without taking the rigid lid assumption. We proceed by attempting to solve the differential equation for the vertical eigenmodes by creating a differentiation matrix on a Chebyshev grid using the methods presented in Trefethen[8].

On the interval $[-1, 1]$, the Chebyshev points on a grid with $N+1$ points take the form:

$$x_j = \cos\left(\frac{j\pi}{N}\right),$$

for $j = 0, \cdots, N$. We transform these points onto our interval $[-1, 0]$ so that they take the form:

$$x_j = \frac{1}{2}\left(\cos\left(\frac{j\pi}{N}\right) - 1\right)$$

We then calculate the differentiation matrix on these points using Trefethen's method, which is to calculate the off diagonal elements via the formula:

$$(D_N)_{ij} = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{x_i - x_j}, \qquad \text{for } i \neq j,$$

where:

$$c_i = \begin{cases} 2 & \text{if } i = 0, N \\ 1 & \text{otherwise.} \end{cases}$$

Here we have indexed our matrix from 0 up to N.

The diagonal elements are then calculated by exploiting the fact that differentiation of a constant function should yield 0 and thus:

$$D_N \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} = 0,$$

so that we have:

$$(D_N)_{ii} = -\sum_{j=0,\, j\neq i}^{N} (D_N)_{ij}.$$

We now need to put this to use in order to solve the equation:

$$(N^{-2}Z_n')' + \lambda_n^2 Z_n = 0,$$

subject to the boundary conditions:

$$Z_n'(-1) = 0 \tag{2.1}$$

$$A^2 Z_n'(0) + Z_n(0) = 0. \tag{2.2}$$

We approach this problem by attempting to convert the problem into a generalised eigenvalue problem for the eigenvalues λ_n and the eigenfunction Z_n. If we start with a Chebyshev differentiation matrix $\widetilde{D}$ on a Chebyshev grid: $0 = z_0, z_1 \cdots z_{N-1}, z_N = -1$, and define $M = \text{diag}(N^{-2}(z_i))$ we will then construct:

$$D = \widetilde{D}M\widetilde{D}, \tag{2.3}$$

which is the matrix corresponding to our operator. However we need to impose the boundary conditions before we can attempt to search for the eigenvalues and eigenvectors of this matrix. We have two choices for this, first we can take our matrix $\widetilde{D}$ and replace the first row with:

$$(-1/A^2, 0, 0, \cdots, 0),$$

which imposes (2.2), and replace the last row with zeros, which imposes (2.1). If we define the matrix formed in this way to be D', then we can search for the eigenvalues and eigenvectors of the matrix:

$$D = \widetilde{D}MD',$$

which will satisfy our eigenvalue problem along with conditions (2.1) and (2.2). The second option, which will be used in the rest of this paper, is to form our matrix via (2.3) and replace the first row with:

$$\widetilde{D} + \frac{1}{A^2}I,$$

where I is the $(N+1) \times (N+1)$ identity matrix. Then replace the last row with $\widetilde{D}$. Finally define C to be $\text{diag}(0, 1, 1, \cdots, 1, 1, 0)$, that is the $(N+1) \times (N+1)$ identity matrix, with the first and last diagonal element set to zero. we then can solve the generalised eigenvalue problem:

$$D\mathbf{v} = -\lambda_n^2 C\mathbf{v}.$$

Either of these methods will produce the same results, which is to give us $N-1$ eigenvalues and eigenvectors for our operator subject to the given boundary conditions. This will give us an estimate for our vertical eigenmodes on our Chebyshev grid which can then be interpolated onto our whole space. The code used for this method can be found in Appendix A.1.

We can test the accuracy of this method in specific cases where we know the analytic solution to our equations. If we consider the rigid lid case where $A \gg 1$ with uniform stratification $N(z) \equiv 1$, then our differential equation becomes:

$$Z_n'' + \lambda_n^2 Z_n = 0,$$

with the boundary conditions:

$$Z_n'(0) = Z_n'(-1) = 0.$$

The solution to this is clearly:

$$Z_n = A_n \cos(n\pi z),$$

where our eigenvalues are $\lambda_n = n\pi$. This example allows us to test the accuracy and validity of our method for solving the differential equation, but has the unfortunate limitation of only being valid in the rigid lid case, where our boundary condition on the free surface has been altered. A plot of the first 9 eigenfunctions, found using the same code with specific choices of our parameters, interpolated using the built in subroutines from Matlab can be found in figure 2.1. We see that although the

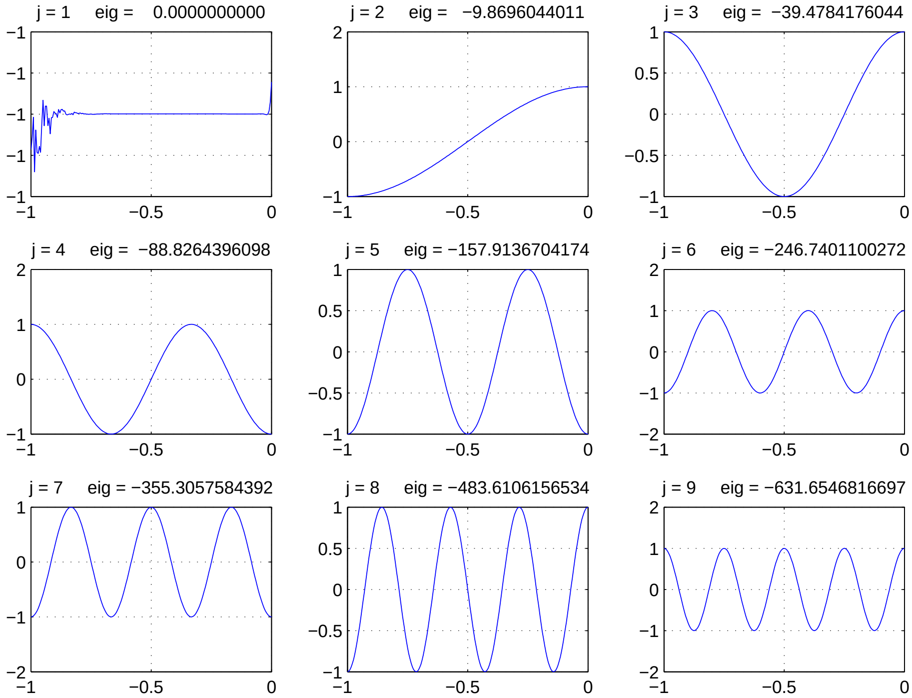

Figure 2.1: Plot of the first 9 eigenfunctions in the uniform stratification, rigid lid case

constant eigenfunction has been computed accurately on the Chebyshev grid, the Matlab interpolation is unstable in this case. We address this by using the barycentric interpolation programs developed by Weideman and Reddy[9], the results of which can be seen in figure 2.2. We compare our results with analytic results using the code in Appendix A.2 and find that the error in the eigenvalues and eigenfunctions on our Chebyshev grid is of order 10^{-11} using only 36 points.

In order to test the validity of our boundary condition we attempt to construct a differential equation for which we have an analytic solution satisfying this boundary condition.

Consider functions of the form:

$$f(z) = e^{\alpha z} \cos(\beta z).$$

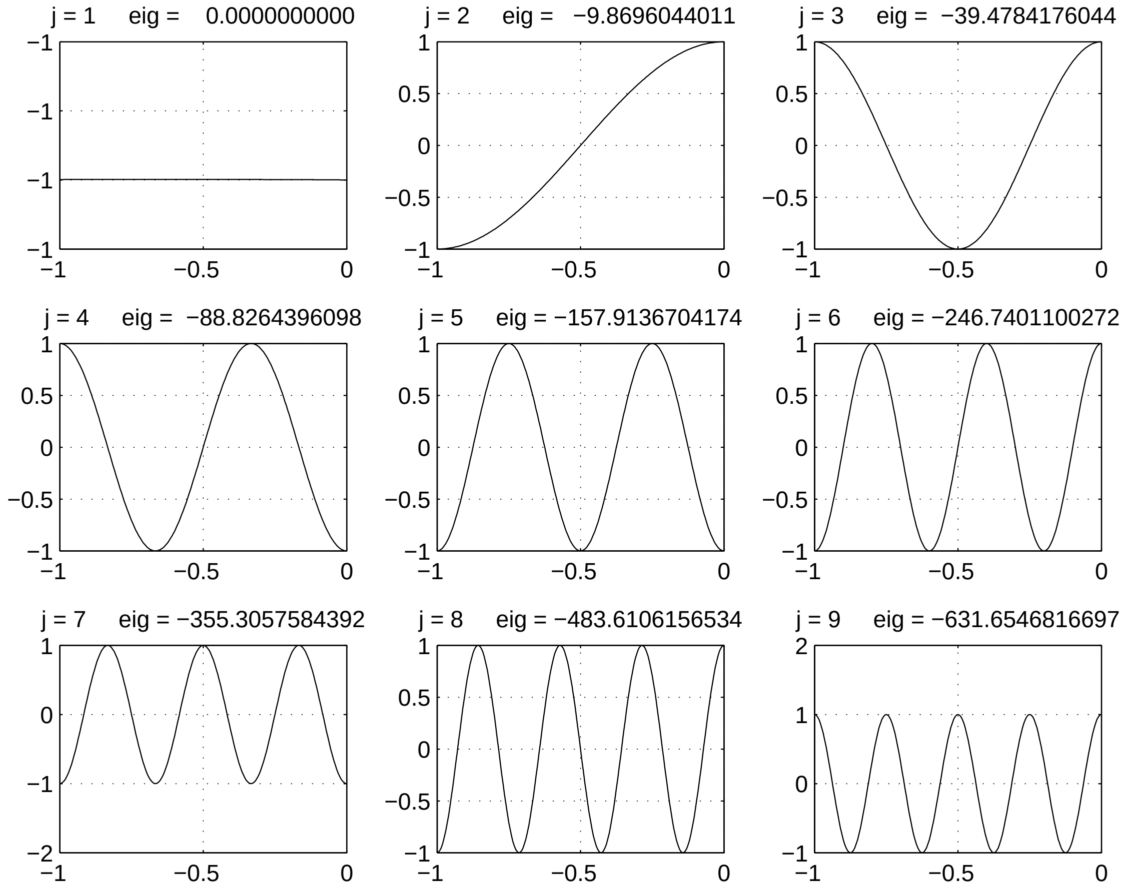

Figure 2.2: Plot of the first 9 eigenfunctions in the uniform stratification, rigid lid case after barycentric interpolation

Then:

$$f'(z) + \frac{1}{A^2} f(z) = e^{\alpha z} \left(\left(\frac{1}{A^2} + \alpha \right) \cos(\beta z) - \beta \sin(\beta z) \right).$$

If we choose $\alpha = -1/A^2$, then $f(z)$ will satisfy the required boundary condition for $z = 0$. Now we need to construct a differential equation satisfied by this function. We have:

$$\begin{aligned} f(z) &= e^{-z/A^2} \cos(\beta z) \\ f'(z) &= -e^{-z/A^2} \left(\frac{1}{A^2} \cos(\beta z) + \beta \sin(\beta z) \right) \\ f''(z) &= e^{-z/A^2} \left(\left(\frac{1}{A^4} - \beta^2 \right) \cos(\beta z) + \frac{2\beta}{A^2} \sin(\beta z) \right). \end{aligned}$$

Combining the second and third equations to get rid of the sin terms:

$$f''(z) + \frac{2}{A^2} f'(z) = -\left(\frac{1}{A^4} + \beta^2 \right) e^{-z/A^2} \cos(\beta z);$$

which leads us to the differential equation:

$$f'' + \frac{2}{A^2} f' + \left(\frac{1}{A^4} + \beta^2 \right) f = 0, \tag{2.4}$$

with the boundary conditions:

$$\begin{aligned} f'(0) + \frac{1}{A^2} f(0) &= 0 \\ f'(-1) &= 0. \end{aligned}$$

The general solution of this equation is:

$$f(z) = e^{-z/A^2} (\widetilde{A} \cos(\beta z) + \widetilde{B} \sin(\beta z)).$$

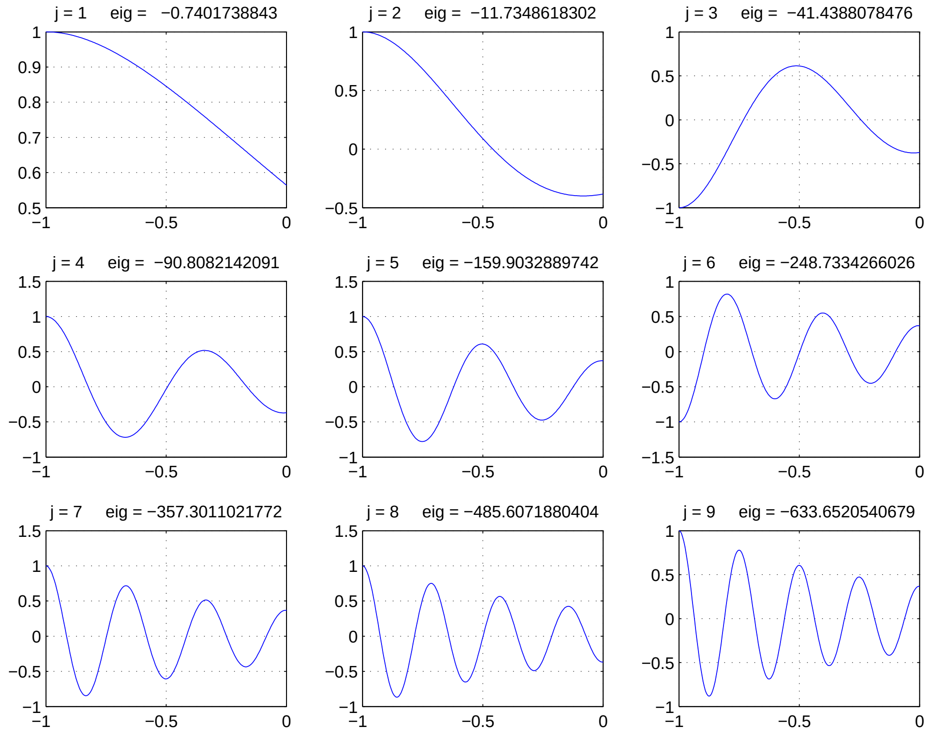

Figure 2.3: Plot of the first 9 eigenfunctions of equation (2.4)

While the boundary conditions imply that $\widetilde{B} = 0$ and $\beta \tan \beta = 1/A^2$. We can then form this into a linear eigenvalue problem with differentiation matrices as we did previously in order to test the application of our free surface boundary condition. For the sake of comparison, we set $A = 1$ and, using the code found in Appendix A.3, we can compare the accuracy of our results with our analytic solutions. We find that the error in the eigenvalues from those computed numerically using built in subroutines from Matlab, as well as the eigenfunction on our Chebyshev points are of the order 10^{-11}, again using only 36 points. A plot of the first 9 eigenfunctions can be found in figure 2.3.

Once we have found our eigenvalues and vertical eigenmodes, we can attempt to construct our series solution for the wave modes by satisfying the shelf boundary conditon (1.23), which we repeat here for clarity:

$$N^2(z)\sum_{n=0}^{\infty} a_n(1-K\lambda_n)e^{-\lambda_n d(z)/B} Z_n(z) = B^{-1}Kd'\sum_{n=0}^{\infty} a_n e^{-\lambda_n d(z)/B} Z_n'(z). \qquad (2.5)$$

We will attempt to collocate this series and cut it off after summing a finite number of terms and satisfy this condition at a collection of points by solving the resulting generalised eigenvalue problem for K and a_n. In the interest of accuracy, we attempt to satisfy the condition on a subset of our Chebyshev grid in order to avoid inaccuracy in polynomial interpolation. We take a conservative estimate of the accuracy of our spectral method and assume that roughly $N/4$ of our eigenvalues and eigenfunctions are reasonably accurate, although in practice we find that considerably more will be accurate enough. Rearranging, we find that:

$$N^2(z_i)\sum_{n=0}^{\lfloor N/4\rfloor} a_n e^{-\lambda_n d(z_i)/B} Z_n(z_i) = K\sum_{n=0}^{\lfloor N/4\rfloor} a_n e^{-\lambda_n d(z_i)/B}(N^2(z_i)\lambda_n Z_n(z_i) + B^{-1}d'(z_i)Z_n'(z_i)),$$

where we have chosen to impose this condition on $\lfloor N/4\rfloor + 1$ points from our Chebyshev grid, i.e. we require this to be true for $i = 4j$, $0 \le j \le \lfloor N/4\rfloor - 1$, and for $i = N$. We therefore define two matrices as:

$$A_{ij} = N^2(z_i)e^{-\lambda_n d(z_i)/B} Z_n(z_i)$$
$$B_{ij} = N^2(z_i)\lambda_n Z_n(z_i) + B^{-1}d'(z_i)Z_n'(Z_i),$$

and solve the generalised eigenvalue problem:

$$A\mathbf{a} = KB\mathbf{a}.$$

This will give us the wave speeds for our modes, K, as well as the coefficients a_n for their expansion in terms of the vertical eigenmodes.

Applying this method, we find the surprising result that although our eigenvalues and eigenmodes are accurate, some of the resulting wave speeds using this method are complex, using only a linearly sloping shelf. The contravenes the basis of our method and is not an appropriate result, but even discounting these as bad eigenvalues, we find that the real eigenvalues do not converge in any sensible way. We investigate this issue by considering the analytic case again and attempt to apply this boundary condition on the shelf using our known analytic functions; we find the same results in this case. The issue is apparently that the exponential term in our series (2.5) decays too rapidly for higher eigenvalues, resulting in an inability to accurately find our wave speeds unless the value of B is taken so large that the shelf is essentially flat. We therefore attempt to solve this problem via a different approach.

Chapter 3

Integral Formulation

We will attempt to reduce our problem from a two dimension differential equation to a generalised eigenvalue problem involving one dimensional line integrals. We begin by returning to our differential equation for p:

$$p_{yy} + B^{-2}(N^{-2}p_z)_z = 0.$$

Setting $Z = Bz$ and $H(y) = Bh(y)$, this becomes:

$$\mathcal{L}p = p_{yy} + (N^{-2}p_Z)_Z = 0,$$

where we have defined $\mathcal{L}$ to be our this weighted Laplacian operator. Let us define a weighted gradient operator ∇_N to be such that $\nabla_N p = p_y\widehat{\mathbf{j}} + N^{-2}p_Z\widehat{\mathbf{k}}$, so that:

$$\mathcal{L}p = \nabla \cdot (\nabla_N p).$$

In order to progress with this method, we will need to consider only the rigid lid case. That is we assume that $a \gg 1$, in order to simplify our boundary conditions. This has the unfortunate result of making it impossible to find the fundamental mode, which requires the flexible lid. But as is shown by Huthnance[3] the other modes are not significantly altered from real world examples of the Rossby radius. Considering $\widehat{\mathbf{n}}$ to be the outward facing normal to our domain, i.e.

$$\widehat{\mathbf{n}} = \begin{cases} \widehat{\mathbf{k}} & \text{if } Z = 0 \\ -\widehat{\mathbf{k}} & \text{if } Z = -B, y \geq 1 \\ -m(y)\left(\widehat{\mathbf{j}} + 1/H'\widehat{\mathbf{k}}\right) & \text{if } Z = -H(y), y \leq 1, \end{cases}$$

where:

$$m(y) = \frac{1}{(1 + (1/H'^2))^{1/2}}.$$

Then our boundary conditions can be written:

$$\begin{aligned} (\nabla_N p) \cdot \widehat{\mathbf{n}} = 0 \quad & \text{for } Z = 0 \text{ or} \\ & Z = -B, y \geq 1 \\ m(y)p - c(\nabla_N p) \cdot \widehat{\mathbf{n}} = 0 \quad & \text{for } Z = -H(y), y \leq 1. \end{aligned}$$

Now suppose that we have a fundamental solution for our operator $\mathcal{L}$, that is $G(y, Z, y_0, Z_0)$ such that:

$$\mathcal{L}G(y, Z, y_0, Z_0) = \pi\delta(y - y_0)\delta(Z - Z_0),$$

where we require that G satisfies our Neumann boundary conditions on the surface and bottom of our domain, but not the boundary condition on the shelf. We consider the fact that $\mathcal{L}f = \nabla \cdot (\nabla_N f)$ and using the identity $\nabla \cdot (\psi\mathbf{A}) = \psi\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\psi$:

$$\begin{aligned} p\mathcal{L}G - G\mathcal{L}p &= p\nabla \cdot (\nabla_N G) - G\nabla \cdot (\nabla_N p) && (3.1) \\ &= \nabla \cdot (p\nabla_N G - G\nabla_N p) - \nabla p \cdot \nabla_N G + \nabla G \cdot \nabla_N p && (3.2) \\ &= \nabla \cdot (p\nabla_N G - G\nabla_N p). && (3.3) \end{aligned}$$

We can then apply Green's lemma, denoting our domain as A with boundary S, we see that:

$$\begin{aligned}\iint_A p\mathcal{L}G - G\mathcal{L}p \, dA &= \iint_A \nabla \cdot (p\nabla_N G - G\nabla_N p) \, dA \\ &= \int_S (p\nabla_N G - G\nabla_N p) \cdot \widehat{\mathbf{n}} \, ds.\end{aligned}$$

We now note that our boundary conditions imply that the line integral on the right hand side vanishes on the surface and bottom, and as we require that the pressure gradient vanishes as $y \to \infty$, we see that the integral reduces to one along the shelf, denoted C. We can then apply our boundary condition on the shelf to rewrite the term involving the weighted normal derivative, $(\nabla_N p) \cdot \widehat{\mathbf{n}}$ in terms of p itself. The right hand side then reduces to:

$$\int_S (p\nabla_N G - G\nabla_N p) \cdot \widehat{\mathbf{n}} \, ds = \int_C p(\nabla_N G) \cdot \widehat{\mathbf{n}} - \frac{m(y)}{c} G p \, ds.$$

We now attempt to analyse the integral on the left hand side. Note first that our governing equation for p is:

$$\mathcal{L}p = 0,$$

and that our governing equation for G is:

$$\mathcal{L}G = \pi\delta(y - y_0)\delta(Z - Z_0),$$

so that the integral on the left reduces to:

$$\iint_A p\mathcal{L}G - G\mathcal{L}p \, dA = \pi p(y_0, Z_0).$$

Rearranging, we find that:

$$\int_C m(y) G p \, ds = c\left(-\pi p(y_0, Z_0) + \int_C p(\nabla_N G) \cdot \widehat{\mathbf{n}} \, ds\right). \tag{3.4}$$

This then becomes a generalised eigenvalue problem:

$$\mathcal{A}p = c\mathcal{B}p,$$

for the eigenvalues, c (the wave speeds), and the eigenfunctions p. We approach solving this problem by reducing it to a linear eigenvalue problem by using Simpson's rule. However, before we proceed, we first need to find the fundamental solution, G. In the uniform stratification case, as was investigated by Schmidt and Johnson[6], the fundamental solution is found to be:

$$\begin{aligned} G(y, Z, y_0, Z_0) =& \frac{1}{2}\log\left|\sinh\left(\frac{\pi}{2B}\left(y - y_0 + i(Z - Z_0)\right)\right)\right. \\ & \left.\times \sinh\left(\frac{\pi}{2B}\left(y - y_0 + i(Z + Z_0)\right)\right)\right| - \frac{\pi}{2B}(y + y_0). \end{aligned}$$

However in the non-uniform stratification case, we must attempt to find this solution in another manner. Schmidt and Johnson[6] state that in this case, the Green's function can be found to be the series:

$$G(y, Z, y_0, Z_0) = \sum_{n=0}^{\infty} a_n \exp(-\lambda_n |y = y_0|) \begin{cases} Z_n(Z)Y_n(Z_0) & \text{if } Z < Z_0 \\ Z_n(Z_0)Y_n(z) & \text{if } Z > Z_0, \end{cases}$$

where Z_n and Y_n are two independent solutions to the equation 3.5. Note here that have in fact used Z_0 to denote the point in Z at which we calculate the Green's function, as well as the zeroth vertical eigenmode, the context in which the notation is used is enough to separate the two however.

We in fact find this is not the correct form for the Green's function, as can be seen by attempting to solve:

$$\mathcal{L}G = \pi\delta(y - y_0)\delta(Z - Z_0).$$

In order to proceed, we first turn to the homogeneous problem:

$$\mathcal{L}f = 0,$$

with the homogeneous Neumann boundary conditions:

$$f_Z = 0 \quad \text{for } Z = 0 \text{ or } Z = -B.$$

We then find, as in Chapter 2, the vertical eigenmodes, which form a complete, orthogonal set satisfying the boundary conditions and homogeneous equation, satisfy:

$$(N^{-2}Z_n')' + \lambda_n^2 Z_n = 0. \tag{3.5}$$

We therefore attempt to find a fundamental solution for out operator $\mathcal{L}$ as a series expansion in these vertical eigenmodes:

$$G(y, Z, y_0, Z_0) = \sum_{n=0}^{\infty} Y_n(y, y_0, Z_0) Z_n(Z).$$

In order to progress, we first need to find the expansion for the Dirac delta function in terms of our vertical eigenmodes:

$$\delta(Z - Z_0) = \sum_{n=0}^{\infty} a_n(Z_0) Z_n(Z).$$

Multiplying both sides by Z_m and integrating from $-B$ to 0, taking advantage of the orthogonality of these eigenmodes, we clearly find that:

$$a_n = \frac{Z_n(Z_0)}{\int_{-B}^{0} Z_n^2 \, dZ}.$$

Our differential equation for G, ignoring the factor of π as we can multiply our result by π a the end to get the result we need, then becomes:

$$\begin{aligned}
\mathcal{L}G(y, Z, y_0, Z_0) &= \delta(y - y_0)\delta(Z - Z_0) \\
\mathcal{L}\left(\sum_{n=0}^{\infty} Y_n(y, y_0) Z_n(Z)\right) &= \delta(y - y_0) \sum_{n=0}^{\infty} \frac{Z_n(Z_0)}{\int_{-B}^{0} Z_n^2 \, dZ} Z_n(Z) \\
\sum_{n=0}^{\infty} \left(Y_n'' Z_n + Y_n (N^{-2} Z_n')'\right) &= \delta(y - y_0) \sum_{n=0}^{\infty} \frac{Z_n(Z_0)}{\int_{-B}^{0} Z_n^2 \, dZ} Z_n(Z) \\
\sum_{n=0}^{\infty} \left(Y_n'' - \lambda_n^2 Y_n\right) Z_n &= \delta(y - y_0) \sum_{n=0}^{\infty} \frac{Z_n(Z_0)}{\int_{-B}^{0} Z_n^2 \, dZ} Z_n(Z).
\end{aligned}$$

Again, taking advantage of the orthogonality of the vertical modes, our equation for Y_n becomes:

$$Y_n'' - \lambda_n^2 Y_n = \frac{Z_n(Z_0)}{\int_{-B}^{0} Z_n^2 \, dZ} \delta(y - y_0).$$

For $\lambda_n > 0$, the solution to this obviously becomes:

$$Y_n = \begin{cases} Ae^{\lambda_n y} + Be^{-\lambda_n y} & \text{if } y > y_0 \\ Ce^{\lambda_n y} + De^{-\lambda_n y} & \text{if } y < y_0, \end{cases}$$

where A, B, C, D depend on the value of y_0. Requiring continuity across the point $y = y_0$ and requiring that there is no exponential growth as $y \to \pm\infty$, the solution becomes:

$$Y_n = Ae^{-\lambda_n |y - y_0|}.$$

The value of A may be determined by the size of the discontinuity in the derivative of Y_n across the point $y = y_0$, found from the coefficient of the Dirac Delta function in the differential equation:

$$\begin{aligned}
Y_n'(y_0^+) - Y_n'(y_0^-) &= \frac{Z_n(Z_0)}{\int_{-B}^{0} Z_n^2 \, dZ} \\
-2A\lambda_n &= \frac{Z_n(Z_0)}{\int_{-B}^{0} Z_n^2 \, dZ}
\end{aligned}$$

$$A = -\frac{Z_n(Z_0)}{2\lambda_n \int_{-B}^{0} Z_n^2 \, dZ}.$$

However, in the rigid lid case:

$$\lambda_0 = 0 \text{ and } Z_0(Z) \equiv 1$$

Where we have obviously chosen the constant vertical eigenmode to have value 1. Our solution for the horizontal mode then becomes:

$$Y_0 = \begin{cases} Ay + B & \text{if } y > y_0 \\ Cy + D & \text{if } y < y_0. \end{cases}$$

Where, again, A, B, C, D may depend on y_0. In order for the Green's function to decay as $y \to \infty$, our solution must be:

$$Y_0 = \begin{cases} 0 & \text{if } y > y_0 \\ A(y - y_0) & \text{if } y < y_0. \end{cases}$$

Taking note of the discontinuity in the derivative, we see that:

$$\begin{aligned} Y_0'(y_0^+) - Y_0'(y_0^-) &= \frac{1}{B} \\ A &= -\frac{1}{B}. \end{aligned}$$

Our complete solution for the Green's function then becomes:

$$G(y, Z, y_0, Z_0) = -\pi \left(H(y_0 - y)\frac{y - y_0}{B} + \sum_{n=1}^{\infty} \frac{Z_n(Z_0)}{2\lambda_n \int_{-B}^{0} Z_n^2 \, dZ} e^{-\lambda_n |y - y_0|} Z_n(Z) \right),$$

where H here denotes the Heaviside Step Function:

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} .$$

We ignore the ambiguity at the point $x = 0$ as it is irrelevant to our problem.

Now that we have calculated a series solution for our Green's function, we may attempt to investigate the computational accuracy of this series. To that end, we may compare the series with the closed form of the Green's function, found in the uniform stratification case. For the sake of simplicity of comparison, let us take:

$$B = 1$$
$$y_0 = 0$$
$$Z_0 = -\frac{1}{2}.$$

In this case, our closed form for the Green's function is:

$$G(y, Z, y_0, Z_0) = \frac{1}{2}\log\left|\sinh\left(\pi\left(y + i\left(Z + \frac{1}{2}\right)\right)\right)\sinh\left(\frac{\pi}{2B}\left(y + i\left(Z - \frac{1}{2}\right)\right)\right)\right| - \frac{\pi}{2}y.$$

While for the series solution, we find that:

$$\lambda_n = n\pi$$
$$Z_n(Z) = \cos(n\pi Z),$$

so we find that our series solution becomes:

$$G(y, Z, y_0, Z_0) = -\pi\left(H(-y)y + \sum_{n=1}^{\infty}\frac{\cos(n\pi/2)}{n\pi}e^{-\lambda_n|y|}\cos(n\pi Z)\right).$$

We may now compare these two forms for the Green's function, and attempt to

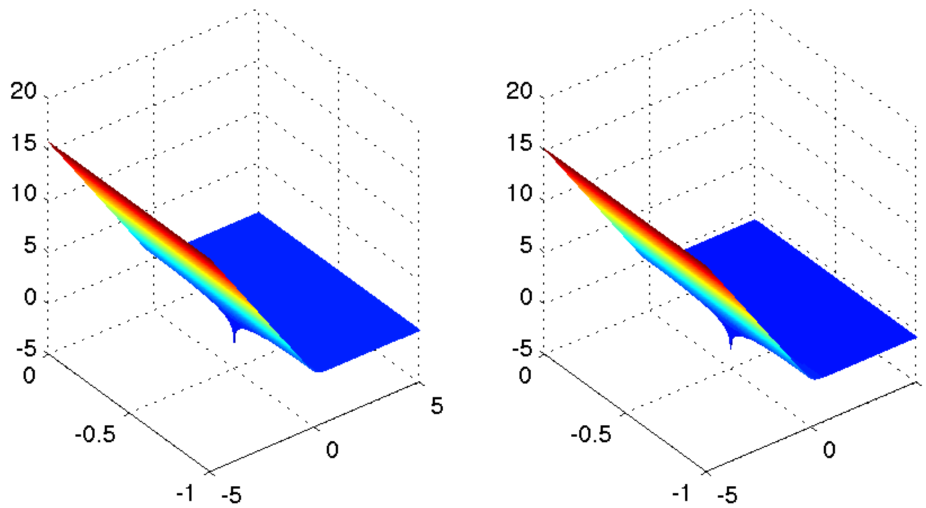

Figure 3.1: Plots of the Green's function using the series solution (left) and the closed form (right)

find the error, as can be seen in figure 3.1. We in fact find that the two forms differ by a constant, $\log(1/2)$, but this is due to the product of two sinh terms within the log in the closed form of the solution, as this produces a factor of $1/4$, which leads to our constant difference. More importantly than this however, we find that there appears to be a Gibb's phenomenon occurring along the line $y = y_0$, which produces a significant error, this can be seen in figure 3.2. This is a serious issue, as the most important behaviour of the Green's function is encapsulated directly around the singularity.

We attempt to combat this problem by applying Fejér's theorem, that is (phrased for our purposes):

Theorem 3.1 The sequence of Cesàro means (arithmetic means) of the sequence of partial sums of a Fourier series of a continuous function converges uniformly.

That is, if s_n is the n^{th} partial sum of our Fourier series for a function f, then:

$$\sigma_n(x) = \frac{1}{n}\sum_{n=1}^{\infty} s_n$$

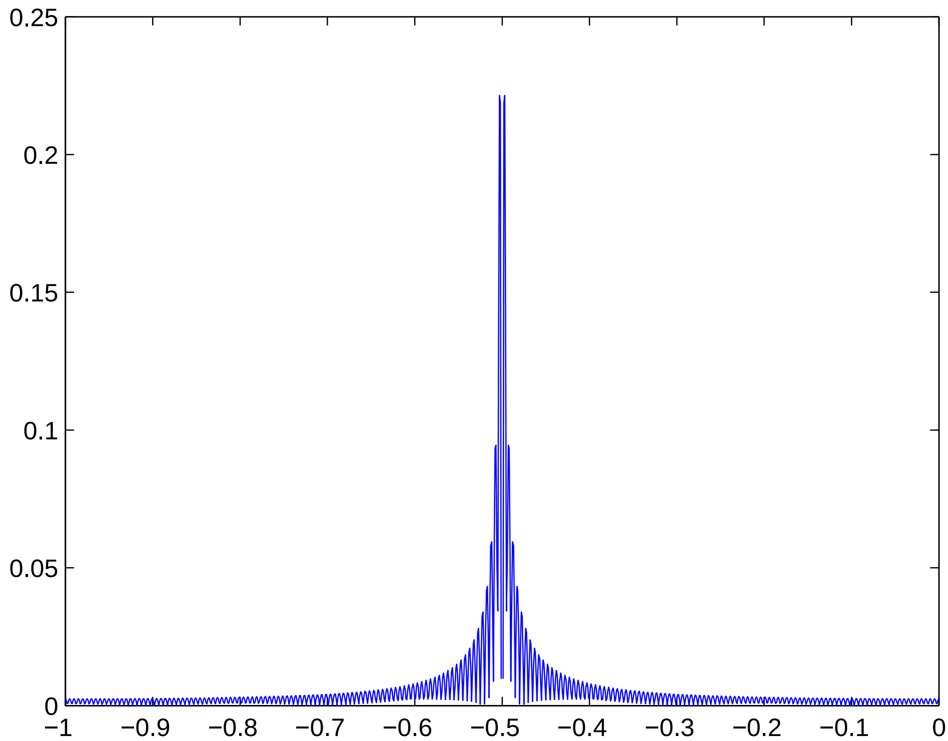

Figure 3.2: The error between our series solution and the closed form along the line $y = y_0$

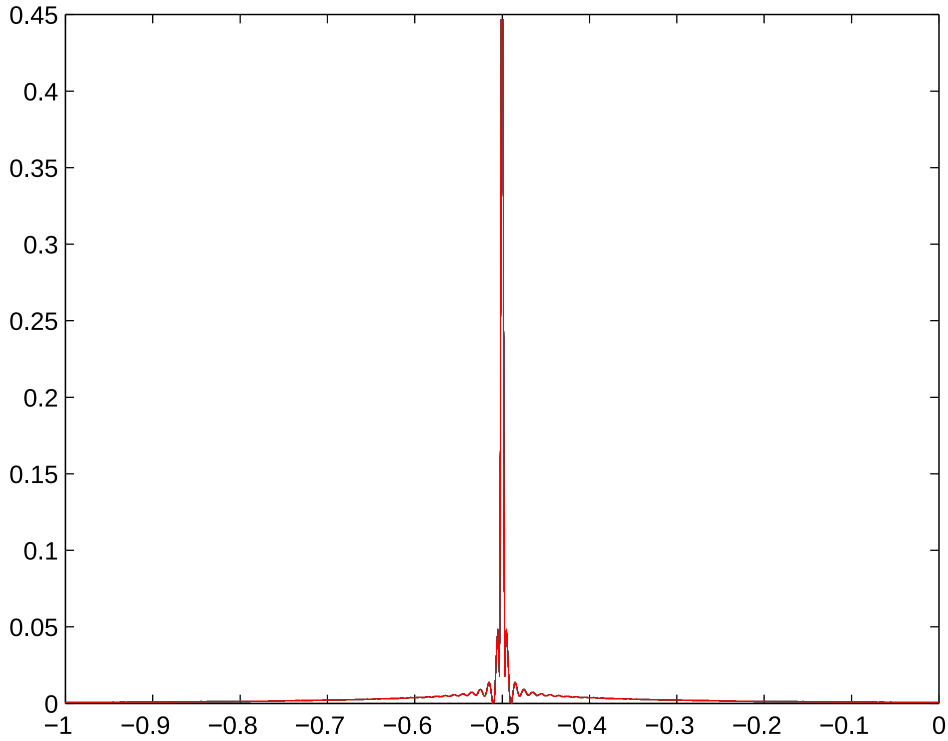

Figure 3.3: The error between our series and the closed form after application of Fejér's theorem and a Shank's transformation along the line $y = y_0$

converges uniformly to f.

The unfortunate result of applying this theorem is that it will slow down the convergence of our series. In order to combat this new complication, we combine the application of this theorem with the application of a Shank's transformation, which will speed up the convergence of our new series.

Taking the same notation for the partial sums as above, the Shank's transformation, S, takes the form:

$$S(s_n) = \frac{s_{n+1}s_{n-1} - s_n^2}{s_{n+1} - 2s_n + s_{n-1}}.$$

Applying these results to our series does in fact result in a much smoother behaviour around the line $y = y_0$, however unfortunately, we still find that there is a significant error along this line, as can be seen in figure 3.3.

We now turn our attention to the accuracy either side of the line $y = y_0$, and attempt to use this to interpolate onto the line. In order to take advantage of the behaviour of this series, we separate the solution into even and odd parts.

$$G(y, Z, y_0, Z_0) = -\pi \left(\sum_{n=1}^{\infty} \frac{Z_n(Z_0)}{2\lambda_n \int_{-B}^{0} Z_n^2 \, dZ} e^{-\lambda_n |y-y_0|} Z_n(Z) - \frac{|y-y_0|}{2B} \right) - \pi \frac{y-y_0}{2B}$$

We attempt to achieve accuracy of the order 10^{-6}, noting that our eigenvalues grow as n, and taking into account exponential decay. To this end we attempt to fit a 5^{th} order polynomial, having taken 200 terms in our series expansion, to find the value of the function on the line $y = y_0$. We may take advantage of the fact that the odd part of this function is known analytically (it is simply linear), so that we only need to interpolate the even part. This means we can fit a 5^{th} order polynomial by only taking 3 points on one side of the line $y = y_0$.

If we take our grid spacing in the y direction to be h, and attempt to fit the polynomial $p(Y) = a_0 + a_1 Y + a_2 Y^2 + a_3 Y^3 + a_4 Y^4 + a_5 Y^5$, where $Y = y - y_0$ and our a_i depend on Z and Z_0, we find:

$$\begin{aligned} a_0 + a_2 h^2 + a_4 h^4 &= G_1 \\ a_0 + 4a_2 h^2 + 16 a_4 h^4 &= G_2 \\ a_0 + 9a_2 h^2 + 81 a_4 h^4 &= G_3, \end{aligned}$$

where:

$$G_i = G(y_0 + ih, Z, y_0, Z_0).$$

We solve this simple linear system for a_0, the interpolated value on the line $y = y_0$ to find:

$$a_0 = \frac{1}{10}(15G_1 - 6G_2 + G_3).$$

This interpolation is much more successful than the application of Fejér's theorem

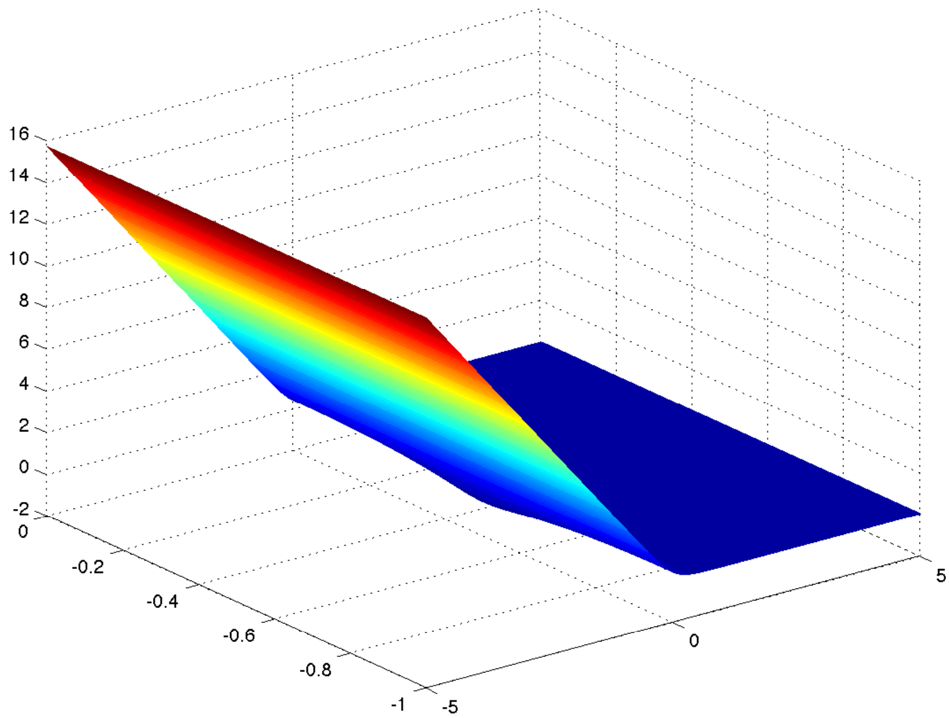

Figure 3.4: Plot of our series solution after interpolation onto the line $y = y_0$

away from the singularity, as is seen in figure 3.5; however close to the singularity, the sharp increase of the function is not captured by this method, and the behaviour of the function is flattened out. This behaviour can be seen in figure 3.4. This issue means that although we have managed to combat the Gibb's phenomenon away from the singularity, we are now presented with a new issue close to the singularity, which is where we need the most accurate results.

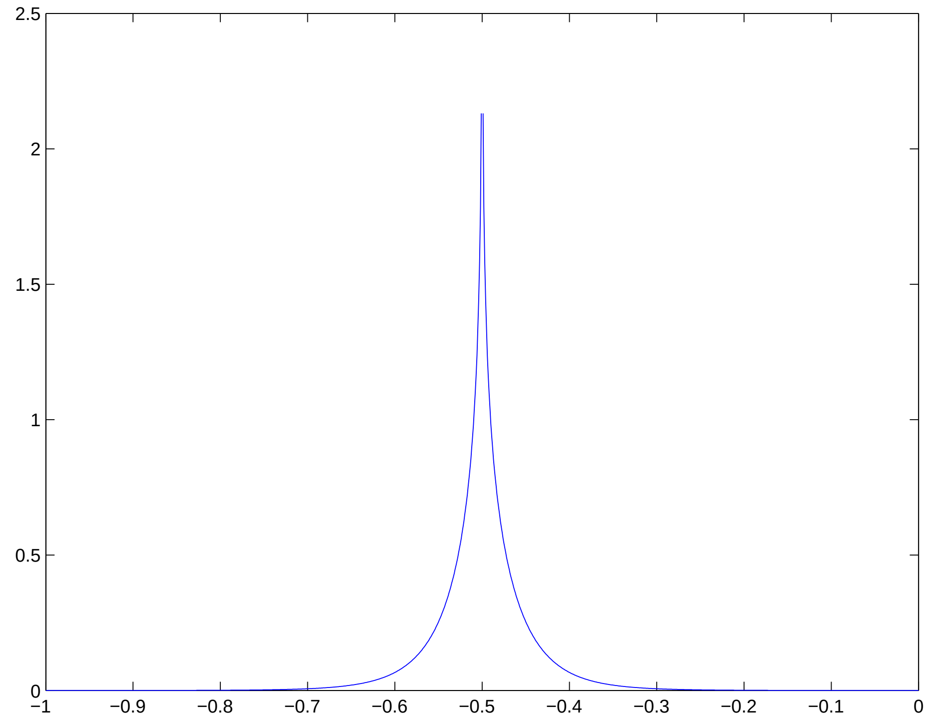

Figure 3.5: The error between our series solution and the closed solution after interpolation onto the line $y = y_0$

We may attempt to combat this issue by noting that close to the singularity, the behaviour is not, in fact, polynomial. We expect that, by considering a Taylor expansion about the point (y_0, Z_0):

$$g(y, Z, y_0, Z_0) = \left|\sinh\left(\frac{\pi}{2B}\left(y - y_0 + i(Z - Z_0)\right)\right)\sinh\left(\frac{\pi}{2B}\left(y - y_0 + i(Z + Z_0)\right)\right)\right|^2$$

will behave as a complex polynomial. Therefore noting the even part of the closed form of the Green's function as:

$$G = \frac{1}{2}\log(g^{1/2}),$$

we expect that $f = \exp(4G)$ should behave polynomially. Therefore, we attempt the same polynomial interpolation on f and then invert the result by applying

$G = 1/4 \log(f)$. The result of this interpolation can be seen in figure 3.6. We now find that even only making steps of one hundredth, we find an error of the order of magnitude 10^{-4}. This error should now be manageably large, as is described in Chapter 4.

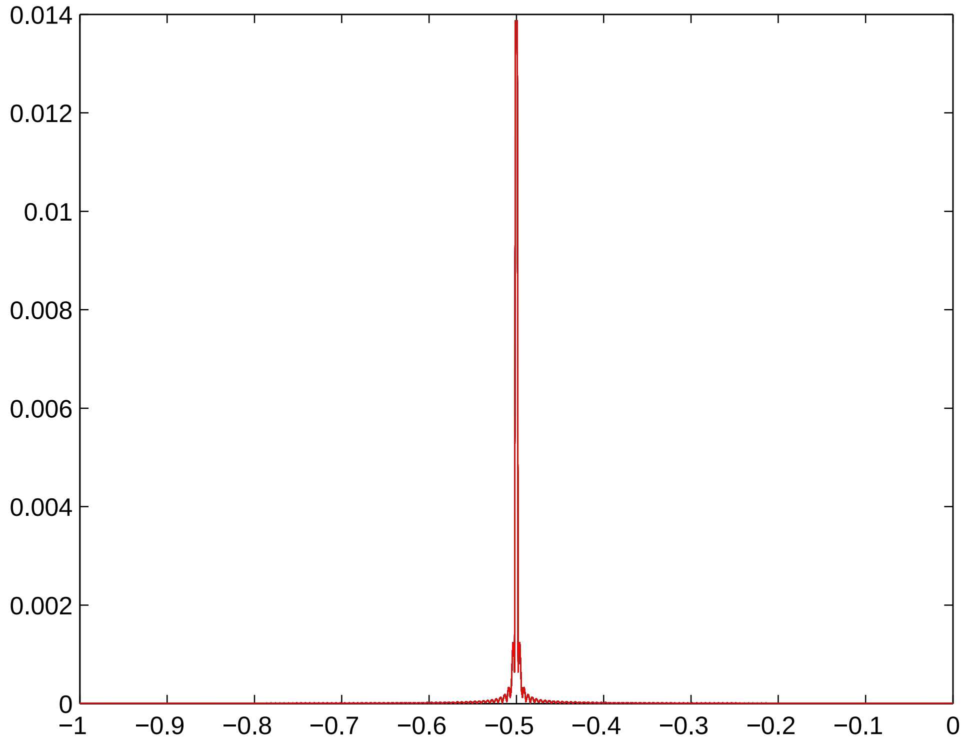

Figure 3.6: A plot of the error between our interpolation of the exponential and the closed form of the Green's function

Now that we find a manageable error, in the series construction of our Green's function, we now combine this construction with the Spectral methods described in Chapter 2, as in the code in Appendix A.5, to find the Green's function for arbitrary stratification. In order to do this, we make use of the Clencurt-Curtis weights, as in Trefethen[8] to integrate our vertical eigenmodes. We can again compare this construction with the closed form of the Green's function for uniform stratification, and find that the error is of the same magnitude as if we use the analytic vertical modes and eigenvalues. This can be seen in figure 3.7.

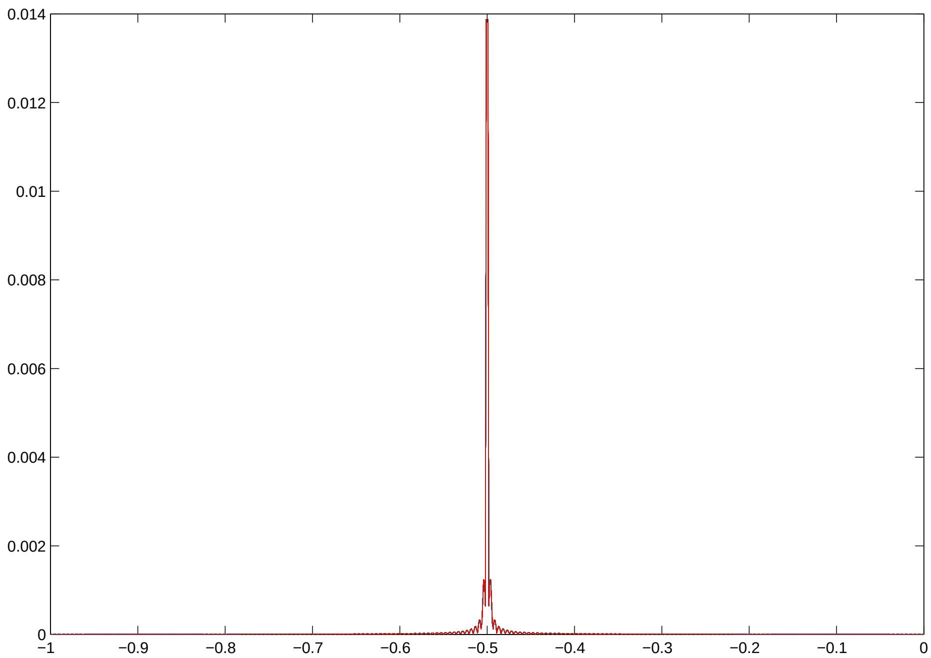

Figure 3.7: A plot of the error between our series calculated using Spectral methods and the closed form of the Green's function

Chapter 4

Conclusion and Further Work

If we turn our attention to the the integral equation 3.4, which we will address as Schmidt and Johnson[6], by removing the singularities and writing it in the form:

$$\int_C m(y)(p - p(y_0, Z_0))G\, ds + p(y_0, Z_0)\int_C Gm(y)\, ds = c\left(\int_C (p - p(y_0, Z_0))(\nabla_N G)\cdot \widehat{\mathbf{n}}\, ds\right).$$

We note that removing the singularities allows us to evaluate these integrals by using Simpson's rule as the integrals are bounded. We see that by removing the singularity from one integral, we have introduced an integral on the left hand side with a singular integrand. This singularity is however integrable, and rewriting the equation in this way ensures that our unknown, p, is only ever multiplied by a finite number. More than this however, it actually improves the expected accuracy of our method, multiplying our Green's function by a function that behaves as $Z - Z_0$ near the singularity along the line $y = y_0$ means that the accuracy of this new function should be expected to be of the order 10^{-6} instead of 10^{-4} as was found to be the error of the Green's function in the previous chapter.

As has already been stated, this integral equation should be addressed by using Simpsons rule, having constructed our Green's function in a series form, using the

Spectral methods described in Chapter 2. In order to maintain accuracy and stability of the method, the vertical eigenmodes have been interpolated using barycentric interpolation in order to interpolate them onto a grid suitable for application of Simpson's rule. The Green's function can then be calculated at the points needed and the integrals evaluated to form a generalised linear eigenvalue problem that can be solved easily to find the wave speeds and pressure fields.

Unfortunately, due to time constraints, this is beyond the scope of this project, and will need to be considered in further work, although the majority of the work has now been completed, as the difficulty of this method lies entirely in the calculation of the Green's function. Once this method has been implemented, it should become the fastest, most accurate and most general method for calculating long waves over arbitrary shelf profiles in arbitrarily stratified coastal seas.

Appendix A

Matlab Code

A.1 Spectral Method Code

```
N =36;
syms y a
f=sym('1');
b=1;
a=0;
  [D,x] = cheb(N);
  D2 = D*diag(subs(f,y,x))*D;
  D1 = D + a*eye(N+1);
  D2(1,:) = D1(1,:);
  D2(N+1,:)= D(N+1,:);
  B=eye(N+1);
  B(1,1)=0;
  B(N+1,N+1)=0;
  [V,Lam] = eig(D2,B);
  lam = diag(Lam);
```

```
  ii= lam<0.1;
  lam=lam(ii);
  Vr=V(:,ii);
  [foo,ii] = sort(-lam);
for j=1:1:9;
  i=ii(j);
  Lambda=lam(i);
  v=Vr(:,i);
  xx = -1:.005:0;
  vv=polint(x,v,xx);
  %vv = polyval(polyfit(x,v,N),xx);
  subplot(3,3,j), plot(xx,vv), grid on
  title(sprintf('j = %d     eig = %15.10f',j,Lambda))
end


xxx=zeros(1,fix(N/4));


for i=1:fix(N/4)-1
    xxx(i)=x(4*i-3);
end
xxx(fix(N/4))=x(N+1);
%vvv = polyval(polyfit(x,v,N),xxx);
M=zeros(length(xxx),length(xxx));
VV=zeros(length(xxx),length(xxx));
DD=zeros(length(xxx),length(xxx));
DD1=zeros(length(xxx),length(xxx));
VV1=zeros(length(xxx),length(xxx));
NN=zeros(length(xxx),length(xxx));
```

```
d=@(x)(-x);
dd=diff(d(y),y);
d1=@(z)(subs(dd,y,z));
L=D*Vr;
for k=1:1:length(xxx);
   M(k,:)=sqrt(-lam(ii(1:length(xxx)))');
   %VV(:,k)=polyval(polyfit(x,V(:,ii(k)),N),xxx);
   %VV1(:,k)=polyval(polyfit(x,D*V(:,ii(k)),N),xxx);
   %VV1(:,k)=polyval(polyder(polyfit(x,D*V(:,ii(k)),N)),xxx);
   for i=1:fix(N/4)-1
   VV(i,k)=Vr(4*i-3,ii(k));
   VV1(i,k)=L(4*i-3,ii(k));
   end
   VV(fix(N/4),k)=Vr(N+1,ii(k));
   VV1(fix(N/4),k)=L(N+1,ii(k));
   DD(:,k)=d(xxx);
   DD1(:,k)=d1(xxx);
   NN(:,k)=subs(1/f,y,xxx);
end
A2=NN.*exp(-(1/b)*M.*DD).*VV;
B=(exp(-(1/b)*M.*DD)).*((1/b)*DD1.*VV1+M.*VV.*NN);
[AA,KK]=eig(A2,B);
K2=diag(KK);
good=K2~=Inf;
K2=K2(good);

[fooo,iii]=sort(-abs(K2));
K=zeros(length(K2),1);
```

```
K(:)=K2(iii(:));
A=zeros(length(K2),length(K2));
for i=1:1:length(K2)
    A(:,i)=AA(:,iii(i));
end
```

A.2 Spectral Method Analytic Test

```
N =36;
syms y a
f=sym('1');
b=1;
a=0;
  [D,x] = cheb(N);
  D2 = D*diag(subs(f,y,x))*D;
  D1 = D + a*eye(N+1);
  D2(1,:) = D1(1,:);
  D2(N+1,:)= D(N+1,:);
  B=eye(N+1);
  B(1,1)=0;
  B(N+1,N+1)=0;
  [V,Lam] = eig(D2,B);
  lam = diag(Lam);
  ii= lam<0.1;
  lam=lam(ii);
  Vr=V(:,ii);
  [foo,ii] = sort(-lam);
```

```
S=zeros(9,1);
for i=1:9
    S(i)=((i-1)*pi)^2;
end
LamErr=lam(ii(1:9))+S;
SS=zeros(length(x),9);
X=zeros(length(x),9);
for j=1:9
    SS(:,j)=(j-1)*pi;
    X(:,j)=x;
end
L=cos(SS.*X);
for j=1:9
   L(:,j)=L(:,j)/L(length(x),j)*sign(Vr(length(x),ii(j)));
end
Verr=L-Vr(:,ii(1:9));
```

A.3 Spectral Method Boundary Condition Test

```
N = 36;
  [D,x] = cheb(N);
  D2 = D^2+2*D+eye(N+1);
  D1 = D + eye(N+1);
  D2(1,:) = D1(1,:);
  D2(N+1,:)= D(N+1,:);
  B=eye(N+1);
  B(1,1)=0;
```

```
  B(N+1,N+1)=0;
[V,Lam] = eig(D2,B);
  lam = diag(Lam);
  ii= lam<0;
  lam=lam(ii);
  Vr=V(:,ii);
  [foo,ii] = sort(-lam);
  S=zeros(9,1);
for j=1:1:9;
  i=ii(j);
  Lambda=lam(i);
  v=Vr(:,i);
  xx = -1:.005:0;
  vv = polyval(polyfit(x,v,N),xx);
  subplot(3,3,j), plot(xx,vv), grid on
  title(sprintf('j = %d     eig = %15.10f',j,Lambda))
  S(j)=(fzero('x*tan(x)-1',sqrt(-Lambda))).^2;
end
LamErr=lam(ii(1:9))+S;
SS=zeros(length(x),9);
X=zeros(length(x),9);
for j=1:9
    SS(:,j)=sqrt(S(j));
    X(:,j)=x;
end
L=exp(-X).*cos(SS.*X);
for j=1:9
   L(:,j)=L(:,j)/L(length(x),j)*sign(Vr(length(x),ii(j)));
```

```
end
Verr=L-Vr(:,ii(1:9));
```

A.4 Green's Function Code

```
yy=-0.5:0.01:0.5;
zz=-1:0.001:0;
[y,z]=meshgrid(yy,zz);
m=200;
A=zeros(length(zz),length(yy),m);
S=zeros(length(zz),length(yy),m);
for n=1:m+1;
A(:,:,n)=-cos(n*pi/2)/(n*pi)*exp(-n*pi*abs(y)).*cos(n*pi*z);
S(:,:,n)=sum(A,3);
end
A1=A(:,:,1:m-1);
A2=A(:,:,1:m);
A3=A(:,:,m+1);
S1=S(:,:,1:m-1);
S2=S(:,:,1:m);
S3=S(:,:,m+1);
C=1/2*log(abs(sinh(pi/2*(y+1i*(z-1/2))).*sinh(pi/2*(y+1i*(z+1/2)))))-pi/2*y;
C1=C+pi/2*y;
F=sum(A2,3)+1/2*abs(y);
%SS=pi*(sum(S,3)/m-heaviside(-y).*y);
%SSS=pi*((sum(S3,3)/(m+1).*sum(S1,3)/(m-1)-(sum(S2,3)/m).^2)./(sum(S3,3)/(m
%+1)-2*sum(S2,3)/m+sum(S1,3)/(m-1))-heaviside(-y).*y);
```

```
i=y==0;
j=zeros(length(zz),length(yy));
k=zeros(length(zz),length(yy));
l=zeros(length(zz),length(yy));
p=zeros(length(zz),length(yy));
for n=5:length(yy)
   j(:,n)=i(:,n-1);
   k(:,n)=i(:,n-2);
   l(:,n)=i(:,n-3);
   p(:,n)=i(:,n-4);
end
j=logical(j);
k=logical(k);
l=logical(l);
p=logical(p);
G=F;
G(i)=1/10*(15*F(j)-6*F(k)+F(l));
C1(i)=1/10*(15*C1(j)-6*C1(k)+C1(l));
%C1(i)=1/4*log(1/10*(15*exp(4*C1(j))-6*exp(4*C1(k))+exp(4*C1(l))));
E=pi*G;
E(i)=1/4*log(1/10*(15*exp(4*E(j))-6*exp(4*E(k))+exp(4*E(l))));
E=(E-pi*y/2);
F=pi*(F-y/2);
D=abs(E-C+log(1/2));
clf
%subplot(1,2,1)
%mesh(y,z,F)
%subplot(1,2,2)
```

```
%mesh(y,z,C)
%subplot(2,2,3)
mesh(y,z,D)
%subplot(2,2,4)
%mesh(y,z,SSS)
```

A.5 Green's Function calculated from vertical modes found using Spectral methods

```
N =400;
syms y a
f=sym('1');
b=1;
  [D,x] = cheb1(N,b);
  D2 = D*diag(subs(f,y,x))*D;
  D1 = D;
  D2(1,:) = D1(1,:);
  D2(N+1,:)= D(N+1,:);
  B=eye(N+1);
  B(1,1)=0;
  B(N+1,N+1)=0;
  [V,Lam] = eig(D2,B);
  lam = diag(Lam);
  ii= lam<0.1;
  lam=lam(ii);
  Vr=V(:,ii);
  [foo,ii] = sort(-lam);
```

```
yy=-0.5:0.01:0.5;
zz=-b:0.001:0;
[y,z]=meshgrid(yy,zz);
z0=-1/2;
y0=0;
m=200;
A=zeros(length(zz),length(yy),m);
S=zeros(length(zz),length(yy),m);
[x1,w]=clencurt(N);
for n=1:m;
    v=Vr(:,ii(n+1));
    v=v*sign(v(1));
    vv=polint(x,v,zz);
    v0=polint(x,v,z0);
    Lambda=sqrt(-lam(ii(n+1)));
    [yl,v]=meshgrid(yy,vv);
    A(:,:,n)=-(v0/(Lambda*b*w*Vr(:,ii(n+1)).^2))*exp(-Lambda*abs(y-y0)).*v;
end
C=1/2*log(abs(sinh(pi/2*(y-y0+1i*(z-z0))).*sinh(pi/2*(y+1i*(z+z0)))))-pi/2*y;
C1=C+pi/2*y;
F=sum(A,3)+1/2*abs(y);
i=y==0;
j=zeros(length(zz),length(yy));
k=zeros(length(zz),length(yy));
l=zeros(length(zz),length(yy));
p=zeros(length(zz),length(yy));
for n=5:length(yy)
   j(:,n)=i(:,n-1);
```

```
   k(:,n)=i(:,n-2);
   l(:,n)=i(:,n-3);
   p(:,n)=i(:,n-4);
end
j=logical(j);
k=logical(k);
l=logical(l);
p=logical(p);
G=F;
G(i)=1/10*(15*F(j)-6*F(k)+F(l));
C1(i)=1/10*(15*C1(j)-6*C1(k)+C1(l));
%C1(i)=1/4*log(1/10*(15*exp(4*C1(j))-6*exp(4*C1(k))+exp(4*C1(l))));
E=pi*G;
E(i)=1/4*log(1/10*(15*exp(4*E(j))-6*exp(4*E(k))+exp(4*E(l))));
E=(E-pi*y/2);
F=pi*(F-y/2);
D=abs(E-C+log(1/2));
clf
%subplot(1,2,1)
%mesh(y,z,F)
%subplot(1,2,2)
%mesh(y,z,C)
%subplot(2,2,3)
mesh(y,z,D)
%subplot(2,2,4)
%mesh(y,z,SSS)
```

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